

On the Completeness of Conflict-Based Search: Temporally-Relative Duplicate Pruning

Thayne T. Walker^{1,2}, Nathan R. Sturtevant³

¹University of Denver, Denver, USA ²Lockheed Martin Corporation, USA

³Department of Computing Science, Alberta Machine Intelligence Institute (Amii),
University of Alberta, Canada

thayne.walker@du.edu, nathanst@ualberta.ca

Abstract

A well-known and often-cited deficiency of the Conflict-Based Search (CBS) algorithm for the multi-agent pathfinding (MAPF) problem is that it is incomplete for problems which have no solution; if no mitigating procedure is run in parallel, CBS will run forever when given an unsolvable problem instance. In this work, we introduce Temporally-Relative Duplicate Pruning (TRDP), a technique for duplicate detection and removal in both classic and continuous-time MAPF domains. TRDP is a simple procedure which closes the long-standing theoretic loophole of incompleteness for CBS by detecting and avoiding the expansion of duplicate states. TRDP is shown both theoretically and empirically to ensure termination without a significant impact on runtime in the majority of problem instances. In certain cases, TRDP is shown to increase performance significantly.

1 Introduction

The objective of multi-agent pathfinding (MAPF) (Stern et al. 2019) is to find paths for multiple agents to move from their current configurations (or states) to goal states such that their respective paths do not conflict at any time. A *conflict* occurs when agents' shapes overlap at the same time. In contrast to "classic" MAPF with actions of unit duration, real world domains often require the use of continuous-time, non-unit duration actions. In this paper, we seek optimal, conflict-free solutions to the MAPF problem in both classic MAPF and continuous-time MAPF, also known as MAPF_R (Walker, Sturtevant, and Felner 2018).

Optimal MAPF solvers are classified into two broad categories: *coupled* and *decoupled*. Coupled algorithms such as multi-agent A* (MA-A*) (Standley 2010), enhanced partial expansion A* (Goldenberg et al. 2014) and M* (Wagner and Choset 2011) solve MAPF problems in a joint state space, where the joint states of all agents are aggregated into a single state. Decoupled algorithms such as CBS (Sharon et al. 2015), ICTS (Sharon et al. 2013), CBICS (Walker et al. 2021), branch-and-cut-and-price (BCP) (Lam et al.

2019) and enhanced variants of these (Gange, Harabor, and Stuckey 2019; Surynek et al. 2016; Li et al. 2019a) solve MAPF problems without aggregating the agents' state spaces together, or by partially aggregating only some of them. In general, decoupled algorithms have a lower practical computational complexity and hence are more popular. CBS is a popular MAPF algorithm. In this paper we focus on the completeness of CBS, but our approach (especially for duplicate detection) is applicable to most coupled and decoupled algorithms for MAPF_R.

A long-standing problem for CBS is that it will run forever if given an unsolvable problem instance (Sharon et al. 2015). The suggested remedy for this is to run a sub-optimal, polynomial-time, complete algorithm (Kornhauser, Miller, and Spirakis 1984; Sajid, Luna, and Bekris 2012; Okumura 2022) in parallel with the main solver. In the case that no solution exists, the polynomial-time algorithm will report this fact, and then CBS can be terminated. But this reliance on a second MAPF algorithm means that CBS is not natively complete. When using CBS for domains other than classic MAPF (e.g., MAPF_R), it may be difficult to find or invent a separate, complete algorithm for the desired domain. At this time, no complete algorithms generally applicable to MAPF_R are known. In summary, there is a need for making CBS *natively* complete.

Temporally-relative duplicate pruning (TRDP) is a novel technique for CBS which can guarantee completeness for classic MAPF and MAPF_Q, a subset of MAPF_R with discrete rational values. TRDP renders otherwise infinite CBS search spaces finite by detecting and eliminating multi-agent "loops". TRDP can be applied to virtually any MAPF domain to guarantee completeness. Importantly, the theory of TRDP resolves a long-standing deficiency of CBS. Furthermore, our empirical results show that TRDP correctly detects and terminates on unsolvable MAPF problem instances while having only a small effect on the runtime in a representative set of (solvable) classic MAPF and MAPF_Q instances.

2 Problem Definition

MAPF was originally defined for a “classic” setting (Stern et al. 2019) where the movements of agents are coordinated on a two dimensional grid, usually represented as a graph $G=(V, E)$. In classic MAPF, edges have a unit time duration and agents occupy a point in space. This paper uses the definition of MAPF_R (Walker, Sturtevant, and Felner 2018), a variant of MAPF for continuous time execution where G is a weighted graph which (unlike grids) may be non-planar, meaning edges may intersect in areas other than vertices. Every vertex $v \in V$ has coordinates in a metric space and every edge $e \in E$ has a positive real-valued edge weight $w(e) \in \mathbb{R}_+$. This includes self-directed edges for wait actions. $\text{MAPF}_Q \subset \text{MAPF}_R$ uses only positive, rational-valued edge weights $w(e) \in \mathbb{Q}_+$. Weights usually represent the times it takes to traverse edges, but cost and time duration can be treated separately. In MAPF there are k agents and each agent is assigned a start and a goal vertex $V_s = \{start_1, \dots, start_k\} \subseteq V$ and $V_g = \{goal_1, \dots, goal_k\} \subseteq V$ such that $start_i \neq start_j$ and $goal_i \neq goal_j$ for all $i \neq j$.

A *solution* to a MAPF_R (MAPF_Q) instance is $\Pi = \{\pi_1, \dots, \pi_k\}$, a set of single-agent *paths* composed of *states*. A state $s = (v, t)$ is composed of a vertex $v \in V$ and a time $t \in \mathbb{R}_+$ ($t \in \mathbb{Q}_+$). A single-agent path is a sequence of $d+1$ states $\pi_i = [s_i^0, \dots, s_i^d]$, where $s_i^0 = (start_i, 0)$ and $s_i^d = (goal_i, t_g)$ where t_g is the time the agent arrives at its goal and all vertices in the path traverse edges in E . Costs and time steps are always monotonically increasing in a path because edge costs are always greater than zero.

Agents have a shape, (e.g., a circle or polygon), which is situated relative to a *reference point* (Li et al. 2019b). Agents move along edges from a vertex to an adjacent vertex, and could use constant or variable velocity in the metric space. A *conflict* happens when two agents perform actions (by either waiting or traversing edges) which results in their shapes overlapping simultaneously. We seek a *feasible solution*, which has no conflicts between any pairs of paths in Π . MAPF solvers typically optimize for minimum makespan or minimum flowtime. Optimization of classic MAPF, MAPF_Q and MAPF_R is NP-hard (Yu and LaValle 2013).

3 Background and Motivation

Decoupled, search-based algorithms for MAPF_R include sub-optimal ones (Silver 2005; Jansen and Sturtevant 2008; Yakovlev and Andreychuk 2017; Cohen et al. 2019) and optimal ones (Walker, Sturtevant, and Felner 2018; Andreychuk et al. 2019; Walker, Sturtevant, and Felner 2020; Walker et al. 2021; Andreychuk et al. 2021; Coppé and Schaus 2022; Walker 2022), but none of them will terminate when no solution exists for the problem instance. Recall that *completeness* means that an algorithm is guaranteed to terminate in a finite amount of time. There are two parts to complete-



Figure 1: An unsolvable MAPF instance where the agents must swap places.

ness (Edelkamp and Schrodli 2011):

1. Termination with a solution if one exists.
2. Termination with \emptyset if a solution does not exist.

The proof for part 1 in classic MAPF and MAPF_R has been shown repeatedly for CBS (Sharon et al. 2015; Li et al. 2019b; Walker, Sturtevant, and Felner 2020). Significantly, part 2 has been shown to be a problem for CBS and other decoupled algorithms (Sharon et al. 2015; Walker 2022). In fact, these algorithms will run forever given an unsolvable problem instance if another complete algorithm is not run in parallel. Even a simple unsolvable instance like the one shown above will cause CBS to run forever.

Because agents can take actions such as waiting or moving back and forth without coming into conflict, the high-level search tree grows infinitely. The reason for this infinite growth is related to the inclusion of time in the state space. Time is typically part of the state space in MAPF_R and decoupled algorithms, but not necessarily coupled classic MAPF algorithms. Adding the time dimension to a finite map like the one in Figure 1 makes the state space infinite since each time step is distinct and agents can wait in place forever. Special attention to duplicate detection is necessary to make the search space finite.

We again emphasize that no complete, polynomial-time solvers for MAPF_R are known at this time. This means that the strategy of running a separate polynomial-time algorithm in parallel to determine solvability of a MAPF_R instance is not currently an option. Figure 2 illustrates a phenomenon that is introduced by MAPF_R , that agents can conflict in ways that are undefined in Classic MAPF. Although the classic MAPF instance with point agents in Figure 2(a) is solvable, the MAPF_R instance in Figure 2(b) is not. Because of this phenomenon (and others related to non-planar graphs and non-unit costs), one cannot simply analyze the graph of a MAPF_R instance, nor run a classic polynomial-time MAPF solver on the graph to determine solvability.

4 Completeness of CBS

Before discussing the completeness of CBS, it is worth mentioning that in popular MAPF benchmark sets (Stern et al. 2019) there are no unsolvable problem instances. Generally speaking, unsolvable instances are often hand-crafted, and (depending on the domain) tend to be very rare. Our purpose here is not to make CBS scale to larger problem instances, but to close the question of completeness by adding functionality to CBS so

that it is *natively* complete for classic MAPF, and identify an approach to completeness adequate for MAPF_R as well.

Aside from duplicate pruning, CBS could be made complete by first computing the theoretical upper bound on the size of the CT for a problem instance, and then forcing CBS to terminate once the CT grows beyond that upper bound.

The total number of unique states in all possible paths of cost C for a single agent in a Classic MAPF instance can not exceed the lesser of C^3 and $|V|C$, where C is the makespan of the lowest-cost solution (Gordon, Filmus, and Salzman 2021). Hence the total number of unique multi-agent states in all possible solutions is bounded by the lesser of C^{3k} and $(|V|C)^k$. A similar bound for MAPF_R does not exist because the number of unique times for states in a path is infinite. However, for MAPF_Q, the bound is $(Cr)^{3k}$ or $(|V|Cr)^k$. Where $r \in \mathbb{Q}_+$ is the inverse resolution of time (e.g., $r=100$ for a resolution of $1/100$). Note, $r=1$ for Classic MAPF.

It has been shown that the makespan, C , for a solvable MAPF instance is $O(|V|^3)$ (Kornhauser, Miller, and Spirakis 1984). Substituting for C , we find that CBS can immediately terminate when the CT reaches a size of $2^{k(|V|r)^4}$. This approach trivially makes CBS complete. However, this simple approach is not practical. For the example instance in Figure 1, where the red agent must swap places with the blue agent, CBS could not terminate safely until generating a CT of size $O(2^{2 \cdot 3^4})$.

This is often worse than running a coupled algorithm like A*, which (with proper duplicate detection) would terminate after $O(|V|^{4k})$ expansions. This is because a coupled algorithm in Classic MAPF doesn't have to keep track of time, just the relative locations of the agents. If A* were to include time in the duplicate detection (as it usually must for continuous-time MAPF), it too would be incomplete. Neither of these bounds (for A* and CBS) are tight because most of the multi-agent state space is not reachable. For the example instance in Figure 1, only 3 unique states would be expanded in total by A* before termination.

The number of high-level expansions with TRDP is finite, but still has the same upper bound for CBS (on a solvable problem instance) without TRDP. However, with TRDP, the actual search space may theoretically be

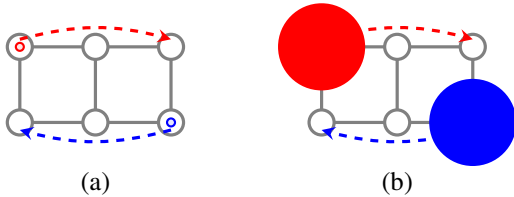


Figure 2: A MAPF instance that becomes unsolvable in MAPF_R with larger agents.

reduced exponentially (Taylor 1997), though one may not see such a reduction in practice. Running CBS with TRDP on the example instance in Figure 1 terminates after only 5 high-level expansions.

5 Mechanics of Duplicate Detection

Duplicate pruning is a technique commonly used with search algorithms in graphs with cycles in order to eliminate exploring sub-optimal paths with loops (Taylor 1997). A *loop* happens when a path visits the same state twice. It is trivial to see that any path with a loop is sub-optimal because concatenating the portions of the path before and after the loop results in a shorter path to the goal.

In MAPF, single-agent loops may be necessary – a feasible solution may correctly contain many agents visiting the same vertex more than once (for example when one agent waits for another), however, *k-agent loops* can be a source of incompleteness. For CBS and many other algorithms that include the time dimension as a part of the state, loops are not recognized by comparing states since revisiting a state (in a multi-agent sense) happens at a different time.

Temporally-relative duplicate pruning (TRDP) defines *temporally-relative duplicate* (TRD) detection which allows the removal of states which have been visited before in a temporally-relative sense.

We define TRD formally as follows: Let $S = \{s_1, \dots, s_k\}$ be a *joint-state* composed of k single-agent states. All single-agent states are not required to have identical times in S , but in our definition, the respective actions taken to arrive at S , must have time overlap. Specifically, for each single-agent action (\hat{s}_i, s_i) for $\hat{s}_i \in \hat{S}$ and $s_i \in S$ where \hat{S} is the ancestor of S , must have time overlap with every other action (\hat{s}_j, s_j) :

$$\hat{s}_i.t \leq s_j.t \leq s_i.t$$

or

$$\hat{s}_j.t \leq s_i.t \leq s_j.t$$

To define a TRD S' of S , we first define $t_{\min}(S)$ to be the earliest single-agent time of S :

$$t_{\min}(S) = \min_{s_i \in S} s_i.t$$

Next, we define a joint-state temporal adjustment function $\Delta_t(S)$, which adjusts the time component of all single-agent states in S to be relative to $t_{\min}(S)$:

$$\Delta_t(S) = \{\forall s_i \in S; (s_i.v, s_i.t - t_{\min}(S))\}$$

S' , a descendant of S , is a TRD of S iff the vertex part of the states are identical and the temporally-relative times are identical. That is: $\Delta_t(S) = \Delta_t(S')$. For example, from Figure 1 we could have a joint state $S = \{(A1, 0.1), (A3, 0.2)\}$. Thus, $t_{\min}(S) = 0.1$, and therefore $\Delta_t(S) = \{(A1, 0.0), (A3, 0.1)\}$. Supposing another state $S' = \{(A1, 0.2), (A3, 0.3)\}$, then $\Delta_t(S') = \{(A1, 0.0), (A3, 0.1)\}$. Because $\Delta_t(S) = \Delta_t(S')$, S' is a TRD of S .

Pruning duplicates from the search space renders a search space finite under certain assumptions (Taylor 1997). However, in MAPF, it is important to ensure that the duplicates are temporally relative. Continuing the previous example, let $S'' = \{(A1, 0.1), (A3, 0.3)\}$. S'' is not a TRD of S ; although the agents are at the same locations ($A1, A3$) respectively, they are not at those locations at the same *relative* time; $\Delta_t(S) \neq \Delta_t(S'')$.

If S'' were part of the only feasible solution Π^* , and S'' were to be pruned (as an erroneous TRD of S), then the algorithm would never find Π^* , making the algorithm incomplete. On the other hand, if S' (being a proper TRD of S) were to be pruned, finding Π^* is not precluded because any path to the goal from S' is identical (in a temporally relative sense) to a path to the goal from S .

6 Duplicate Pruning in CBS

Our discussion will now focus on the implementation of TRDP in CBS (Sharon et al. 2015). First, we explain the CBS algorithm, then we explain the TRDP implementation. Pseudocode for the CBS algorithm is shown in Algorithm 1 with changes for TRDP on line 6.

6.1 CBS

CBS searches a conflict tree (CT) where each node N contains a solution $N.II$. The root node contains paths for each agent without taking the paths of other agents into account (line 3). $N.II$ may contain conflicts, and CBS will detect conflicts between the paths in $N.II$ (line 8). If any conflict exists between the paths for agents i and j , two child nodes N_i and N_j are generated, one for each agent in conflict. Each child node contains a new constraint for a single agent to avoid the conflict (line 15).

Many different types of constraints are possible. For example, a vertex constraint $c = \langle i, v, t \rangle$ blocks agent i from entering vertex v at time t . Taking constraints for N_i and all its ancestors into account (line 14), $\pi_i \in N_i.II$ is re-planned (line 16) to respect the new constraints (and consequently avoid the conflict with agent j). This process is done analogously for agent j . In any case that the agent cannot reach its goal in the context of all constraints (i.e., over-constrained) the CT node is pruned (line 17). Eventually, after enough constraints are accumulated, CBS will find a feasible solution iff one exists.

A partial example of a CT is shown in Figure 4. The root node contains shortest paths to the goal and child nodes resolve conflicts by adding constraints to one agent or the other, which in this case causes the agents to wait. In the figure, an arrow (\leftarrow) means an action that moves the agent to an adjacent node, and a looped arrow (\circlearrowleft) means a wait action. A wait action at the goal is shown with a box around it. The pruning shown with 'X's in the figure is due to TRDP and will be discussed later. The search stops when a conflict-free solution is found in the node with a green border.

6.2 CBS+TRDP

As previously mentioned, any solution containing a k -agent loop is sub-optimal. Since CBS is an optimal algorithm, a solution containing a k -agent loop in a CT node can be treated similar to an infeasible solution. Therefore, TRDP introduces a new type of conflict called a temporally-relative duplicate conflict or *TRD conflict*. A TRD conflict occurs when a temporally-relative k -agent loop exists in a solution $N.II$. Thus, in addition to detecting traditional motion conflicts between pairs of agents (see Algorithm 1 line 8), CBS+TRDP detects TRD conflicts (Algorithm 1 line 6) using the approach defined in the previous section (Algorithm 2).

In response to a TRD conflict, TRDP generates k constraints, one for each agent, which causes CBS to generate k child nodes N_1, \dots, N_k , such that each child node N_i contains a TRD constraint. A *TRD constraint* is a tuple $\langle i, [t_{start}, t_{end}], t_{offset} \rangle$, where i is the agent number, $[t_{start}, t_{end}]$ is a time range in which states are considered to be a *loop-start candidate* state, and t_{offset} is an exact time offset from a loop-start candidate. For example, a TRD constraint of: $\langle i, [0.1, 0.3], 0.4 \rangle$, means that for agent i , any state in the time range $[0.1, 0.3]$ will be considered a loop-start candidate, and any descendant state at the same vertex with an exact time offset of 0.4 is considered a duplicate.

Thus, a TRD constraint enforces the rule that the agent being re-planned at the low level will avoid *all* states s' , which are duplicates of *any* loop-start candidate s , such that $s.t \in [t_{start}, t_{end}]$, $s.v = s'.v$ and $s'.t = s.t + t_{offset}$. For example, during low-level planning, given a TRD constraint of: $\langle i, [0.1, 0.3], 0.4 \rangle$, if the agent's path were to visit the loop-start candidate $s = (v, 0.2)$, it would not be allowed to visit $s' = (v, 0.6)$ because it visits the same vertex at the exact offset time.

The expansion routine for the low-level A* search is shown in Algorithm 3. During the expansion, for any loop-start candidate, we add a *TRD list* to search nodes that contains a list of *TRD pairs* (line 18). Each TRD

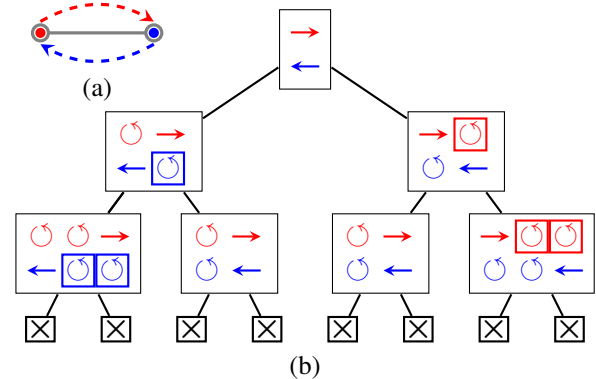


Figure 3: (a) An unsolvable MAPF instance and (b) a partial CBS high-level search tree with TRDP.

Algorithm 1 CBS+TRDP Algorithm

```

1: Input: a MAPF instance
2:  $OPEN \leftarrow \emptyset$ 
3: Initialize the root node and add it to  $OPEN$ 
4: while  $OPEN \neq \emptyset$  do
5:    $N \leftarrow OPEN.pop()$ 
6:    $C \leftarrow \text{FINDTRDs}(N, \Pi)$   $\triangleright$  Find TRDs, return constraints
7:   if  $C = \emptyset$  then
8:      $C \leftarrow \text{FINDCONFLICT}(N, \Pi)$   $\triangleright$  Find conflicts, return
       constraints
9:   end if
10:  if  $C = \emptyset$  then
11:    return  $N, \Pi$ 
12:  else
13:    for  $i, c \in C$  do  $\triangleright$  For each agent  $i$  and set of constraints  $c$ 
14:       $N' \leftarrow N$ 
15:       $N'.C \leftarrow N'.C \cup c$ 
16:       $\pi' \leftarrow \text{Replan}(i, N'.C)$ 
17:      if  $\pi' \neq \emptyset$  then
18:         $N'.\Pi_i \leftarrow \pi'$ 
19:         $OPEN \leftarrow OPEN \cup N'$ 
20:      end if
21:    end for
22:  end if
23: end while
24: return "No solution"

```

Algorithm 2 FINDTRDs: TRD Detection and Constraint Generation

```

1: Input: a MAPF solution  $\Pi$ .
2: Check if a loop  $\langle s, s' \rangle$  occurs in any  $\pi_i \in \Pi$ ; save pointers to
  loops.
3: If fewer than  $k$  paths have loops, return  $\emptyset$ .
4: Check for TRDs  $\langle S, S' \rangle$  using pointers to loops from step 2.
5: If no  $k$ -agent loop exists, return  $\emptyset$ .
6: Create  $k$  TRD constraints for each agent:
7: for  $i \in 1..k$  do
8:    $c_i \leftarrow \langle i, [t_{\min}(S), t_{\max}(S)], t_{\min}(S') - t_{\min}(S) \rangle$ 
9: end for
10: return  $\{c_1, \dots, c_k\}$ , the set of TRD constraints, one for each
  agent.

```

pair contains a pointer to an ancestor node, s , and an offset time $d.offset$. Once set, the TRD list is propagated to descendants.

The regular CBS constraint pruning check is carried out first (line 7). Then the TRD constraint pruning check is carried out (line 11). In this check, if the vertex is the same as some loop-start candidate, and the offset time criteria is exactly met, then the node s' is not generated. Note that s' could still be generated if reached by another path through an ancestor that is not a loop-start candidate.

A* normally contains a dominance check which replaces an open node with any node of lower cost. This is still part of our expansion routine. However if a successor is visited via another path with equal g-cost, but the node has no TRDlist, the TRD list is updated (line 27). This effectively removes the TRDlist and ensures that if any equally low-cost path exists which avoids the loop, it will be retained.

TRDP can be implemented in CBS by modifying the expansion routine as shown in Algorithm 1. The only change is on Line 6 which adds a TRD conflict check before the regular conflict check. If a TRD conflict is found, a node is generated for each agent with appro-

Algorithm 3 Low Level A* Expansion Routine

```

1: Input:  $n$  a single-agent search node,
2:    $C$  a set of "regular" CBS constraints,
3:    $D$  a set of TRD constraints
4:  $\triangleright$  Generate successor nodes.
5: for  $s' \in \text{SUCCESSORS}(n, s)$  do
6:    $\triangleright$  Prune successors that violate regular CBS constraints.
7:   if any  $c \in C$  blocks  $s'$  then
8:     Back to top of loop on line 5.  $\triangleright$  Successor is pruned.
9:   end if
10:   $\triangleright$  Prune successors that would complete a multi-agent loop.
11:  for TRDpair  $\in n.\text{TRDlist}$  do
12:    if TRDpair. $s.v = s'.v$  AND
13:      TRDpair. $s.t + \text{TRDpair.offset} = s'.t$  then
14:      Back to top of loop on line 5  $\triangleright$  Successor is pruned.
15:    end if
16:  end for
17:   $n' \leftarrow \{s', n.gcost + cost(s, s'), n.\text{TRDlist}\}$   $\triangleright$  Successor
  node.
18:   $\triangleright$  Add TRDpairs if applicable.
19:  for  $d \in D$  do
20:    if  $s'.t \geq d.t_{\text{start}}$  AND  $s'.t \leq d.t_{\text{end}}$  then
21:       $n'.\text{TRDlist} \leftarrow n'.\text{TRDlist} \cup \{s', d.offset\}$   $\triangleright$  Add
  TRDpair.
22:    end if
23:  end for
24:   $\triangleright$  Check for dominance.
25:  if  $n' \in OPEN$  AND
26:    ( $n'.gcost < OPEN.fetch(n').gcost$  OR
27:    ( $n'.gcost = OPEN.fetch(n').gcost$  AND  $n'.\text{TRDlist} = \emptyset$ ))
  then
28:     $OPEN.fetch(n') \leftarrow n'$   $\triangleright$  Update OPEN node.
29:  else
30:     $OPEN \leftarrow OPEN \cup n'$ 
31:  end if
32: end for

```

priate TRD constraints to help it avoid the k -agent loop.

The TRD detection and constraint generation algorithm is shown in Algorithm 2. First, each path $\pi \in \Pi$ is checked for a single-agent loop (line 2). If all k paths contain a loop, then all paths are checked for loops which have time overlap (line 4). Finally, TRD constraints are created for each agent with the appropriate information.

Given the MAPF instance in Figure 3(a), Figure 3(b) shows the resulting CT when TRDP is used. In the root node, each agent's path moves straight to its goal. After

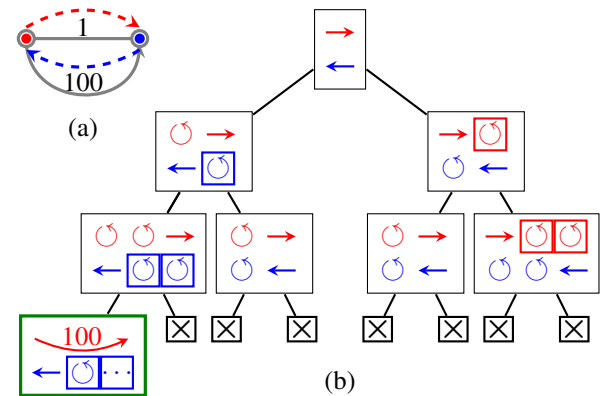


Figure 4: (a) A solvable MAPF_R instance and (b) a partial CBS high-level search tree with TRDP.

the first split, the red agent is forced to wait on the first time step in the left child and the blue agent is forced to wait on the first time step in the right child. Wait actions at the goal are taken into account in the TRD check. TRD constraints are added to each of the four leaf nodes because they each contain a 2-agent loop. Because the TRD constraint in each case blocks the agent from entering the loop, and the regular constraint caused by the split blocks the agent from the conflict, the result is that the low-level solver cannot find a path to the goal and therefore the CT node is pruned. Because of this, CBS correctly terminates with \emptyset .

Figure 4(a) adds a higher-cost directed edge to the instance in 3(a) so that the instance is solvable. In Figure 4(b) the red agent is allowed to take the longer edge in the leftmost leaf node and achieve the goal. This goal is found quickly because moving on the shorter edge is blocked by regular constraints and waiting is blocked by the TRD constraint. Without TRDP, an exponential number of nodes would have to be generated before the cost could increase enough for CBS to accept the case of the red agent taking the longer edge as the optimal solution.

7 Theoretical Analysis

We now show that CBS+TRDP is optimal and complete. First, we show that k -agent loops are sub-optimal. Second, resolving them using TRD constraints can never block an optimal, feasible solution. Finally, that if the problem instance is unsolvable, TRDP guarantees termination.

Definition 1. Over-constrained: A CT node is over-constrained when a solution cannot be found in its sub-tree due to a constraint or collection of constraints which blocks a solution.

Definition 2. Loop: A loop occurs when a single agent visits the same vertex more than once in its path. This includes waiting in place for one or more actions.

Lemma 3. *An optimal path cannot contain a loop.*

Proof. An optimal path cannot contain a loop because a shorter path can always be obtained by concatenating the sub-path before the loop to the sub-path after the loop. \square

Corollary 4. *By extension of Lemma 3, if a solution has a temporally-relative k -agent loop, the solution must be sub-optimal.*

Now that we have shown that TRDs are sub-optimal, we show that pruning them with TRDP never precludes an optimal solution.

Lemma 5. *CBS+TRDP only eliminates sub-optimal or infeasible solutions.*

Proof. Let N be a CT node for which $N.II$ contains a k -agent loop. $N.II$ is sub-optimal per Lemma 3. Consider all optimal solutions which can be reached on a path in the CT:

1. There is no path through N to an optimal solution.
2. There is a path through N to an optimal solution.

For case (1), either some agents are over-constrained in N , or there is no solution to the problem instance in general. Since an optimal solution is not reachable through N , any actions blocked by c_i have no effect on finding an optimal solution in the sub-tree of N .

The proof for case (2) is very similar to the original proof for CBS. In this case, it is certain that at least one of the single-agent loops must be avoided in an optimal solution, thus at least one c_i added to one of the child nodes N_i in the k -way split (based on the k TRD constraints generated in Algorithm 1 line 6) must be correct. If any c_i is incorrect (blocks an optimal solution) then that solution is guaranteed to be found in at least one sibling node of N_i . By contradiction, if the optimal solution is blocked in every sub-tree, then either N is already over-constrained (contradicting our assumptions) or a TRD can exist in an optimal solution, but this violates Lemma 3.

In summary, TRDP either prunes unsolvable sub-trees, or the optimal solution is guaranteed to lie in at least one sub-tree of the child nodes of N . \square

Next, we show that TRDP guarantees termination.

Lemma 6. *TRDP renders the search space of CBS finite for Classic MAPF and MAPF_Q.*

Proof. Let E , the set of edges of G , be finite. Let $W \in \mathbb{Q}$ be the set of edge weights for E . W must be finite. Recall that duplicate detection is performed based on a transformed form of S via the function $\Delta_t(S)$. This is done by subtracting $t_{min}(S)$ from the time ($s.t$) of each single-agent state in S . The range of $t_{min}(S)$ consists of combinations of multiples of W , hence the maximum number of unique times in the range of $\Delta_t(S)$ is $O(\text{MAX}(W)/\text{GCD}(W))$, where GCD is the greatest common denominator of floating point numbers. Because $W \in \mathbb{Q}$, GCD(W) is well-defined and the range of times for $\Delta_t(S)$ is finite. Since the range of times for $\Delta_t(S)$ is finite, and V is finite, the range of $\Delta_t(S)$ is finite.

In summary, although the domain of S is infinite because there is no upper bound on the domain of $s.t$, CBS+TRDP performs duplicate detection on $\Delta_t(S)$ which has a finite range, therefore the search space of CBS+TRDP is finite. \square

Finally, we combine Lemmas 5 and 6 to show that CBS+TRDP is optimal and complete.

Theorem 7. *CBS+TRDP is optimal and complete.*

Proof. Per Lemma 5, CBS+TRDP only eliminates sub-optimal and infeasible solutions. Since CBS searches the CT in a best-first fashion, and optimal solutions cannot be eliminated by TRDP, CBS+TRDP is optimal. Per Lemma 6, TRDP makes the search space finite. Therefore, the size of the CT has a finite upper bound. Hence,

CBS+TRDP is guaranteed to terminate, regardless of whether the problem instance is solvable. \square

8 Bypassing TRD Splits

In settings where the number of agents is large compared to the size of the graph (agent-dense settings), many TRDs can occur, resulting in many k -way splits in the CT. This can lead to a very large CT and cause a significant amount of work. Our experiments showed that TRDP often reduces the depth in the CT at which a solution is found when compared to regular CBS in agent-dense settings. However, the average branching factor is increased. Thus TRDP usually causes a significantly larger CT to be generated when compared with regular CBS.

Fortunately, the k -way split can be completely avoided in many cases. The procedure to do so is simple and is based on the bypass procedure for regular CBS (Boyarski et al. 2015). When a TRD is found, we test each agent individually for a bypass by adding a TRD constraint and re-planning its path. If the agent is able to find an alternate path that (1) avoids the loop, (2) does not increase the path cost, and (3) does not incur more conflicts, this path is a *bypass*. Upon finding a bypass, we replace $\pi_i \in N.\Pi$ with the bypass in the parent node N . Then N is re-inserted into the OPEN list. If we fail to find a bypass, the results of this computation (each c_i and corresponding re-planned paths) are used to generate the k successor nodes.

We found that this simple procedure, on average, avoided up to 99% of k -way splits for agent-dense, solvable instances. We now prove that the TRDP bypass procedure preserves optimality and completeness.

Theorem 8. *CBS+TRDP with the bypass procedure is optimal and complete.*

Proof. Let N be a CT node containing a TRD in $N.\Pi$. If a bypass is found, $N.\Pi$ is fixed with a new π_i to avoid the TRD to create N' , and N' is inserted into OPEN. Per the bypass procedure, the cost of $N'.$ Π is not increased nor decreased, hence does not change the optimality properties of CBS+TRDP. No constraints are added to N' , hence it is impossible to block any optimal or feasible solution in the sub-tree of N' . Therefore, the completeness properties of CBS+TRDP are preserved. In summary, adding the bypass procedure to CBS+TRDP retains optimality and completeness as shown in Theorem 7. \square

9 Empirical Results

We preface this section by reminding the reader that the primary intent of TRDP is to make CBS natively complete, not necessarily to scale to larger problems. Depending on the domain, the probability of encountering an unsolvable instance may be low. (None of the MAPF benchmark set contain unsolvable problem instances.) Additionally, the probability of encountering k -agent loops during the search may be low. In our

investigation, we found that the probability of encountering TRDs during the search is rare in typical problem instances. When running all grid MAPF benchmark problems (Stern et al. 2019), less than 0.01% of problem instances triggered the TRDP logic. The most common encounter of TRDs was in small mazes with corridor widths of 1 (an agent-dense setting).

Fortunately, although problem instances which actually exercise TRDP logic may be relatively rare in practice, its inclusion in CBS has many benefits. Foremost, it ensures completeness for CBS in both MAPF and MAPF_Q settings. Second, TRDP has the ability to prune portions of the high-level search which are unsolvable. Finally, including TRDP it is relatively inexpensive. The complexity of detecting TRDs in a solution is $O(kC^2)$ where C is the makespan of a solution. The expense is normally mitigated further because the occurrence of single-agent loops is also relatively rare, meaning the TRD check usually exits early (see Algorithm 2 line 3 and line 5).

In this section we show two things: (1) that TRDP has performance benefits for hand-crafted examples with high agent density and (2) that TRDP incurs no significant cost for MAPF instances in general.

We ran CBS with TRDP for the entire set of MAPF benchmarks on 4-, 8-, 16- and 32-neighbor grid domains (Rivera, Hernández, and Baier 2017). Note that 4-neighbor grids are classic MAPF instances, but the other domains with higher connectivity are MAPF_Q instances. Our tests start with two agents and increase the number of agents by one until the problem instance is no longer solvable in under 30 seconds. We recorded the runtimes and the total number of agents for which we could solve within the 30 second time limit. Additionally, we counted the number of times that TRDs occurred in any CT node in this process.

Over the nearly 32,000 experiments and millions of CT nodes generated, only 29 TRDs were ever encountered. Most TRDs were found to occur in settings with narrow corridors. These have high agent density per the

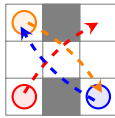
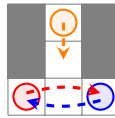
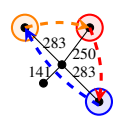
Problem			
No TRDP	309	24	511
With TRDP	297	21	298

Table 1: Size of CT with and without TRDP.


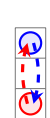
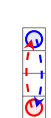
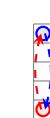
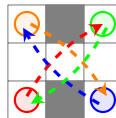
Problem					
CT Nodes	5	36	541	27K	357K

Table 2: Size of CT for unsolvable MAPF instances.

number of traversable edges. As the ratio of agents per graph vertices and edges increases, the probability of encountering TRDs increases. Thus, part of our focus in this section is on agent-dense problem instances.

For some instances with high agent density, using TRDP can help prune parts of the CT in CBS which contain k -agent loops. A worked-out example of such a scenario is shown in Figure 4. Table 2 shows the reduction in CT nodes required to solve some hand-crafted problem instances. The first two instances are for unit-cost grids, and the last instance is for a weighted graph with fixed wait actions of 20. We see that in all three instances, some nodes are pruned from the CT, saving work before the solution is found.

Table 3 shows the amount of total runtime and the amount of runtime attributed to TRD checking when varying the map type, number of agents and branching factor. The top half of the table is for “classic” 4-neighbor grids, and the bottom half of the table is for 32-neighbor grids. Note that in all cases, the exact same solution was found with TRDP turned on, and TRDP did not prune any nodes from the CT. We see a trend that TRDP increases the runtime in most cases, but the increase is less than .01% on average for 4-neighbor grids and less than about .01% on average for 32-neighbor grids. But the overhead was up to about 3% in the most extreme cases. We see that the amount of increase is positively correlated to (1) the number of agents k , (2) the map size (which results in larger mean values for C) and (3) the single-agent branching factor. In the larger branching factor settings (e.g., in 32-neighbor grids) many edges are longer, typically resulting in longer duration actions. This causes TRD checking to take longer because of actions having partial time overlap. Additionally, edge-edge crossings in these graphs are abundant, causing more conflicts in general, resulting in a larger CT. In general, as the size of the CT increases, agents have more constraints, causing longer paths, increasing C , and increasing the number of single-agent loops. These factors cause the TRD check time to increase.

10 Conclusion

We have introduced temporally-relative duplicate pruning (TRDP), a technique for decoupled MAPF and MAPF_Q algorithms for guaranteeing completeness. TRDP solves a long-standing problem for CBS by making CBS natively complete. We have also shown theoretically and empirically that TRDP has desirable properties for MAPF algorithms, namely, increased efficiency for agent-dense settings and completeness guarantees. TRDP adds these benefits while adding no significant computational overhead.

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4-Neighbor Grid					
Map	Size	4 Agents		10 Agents	
		Total	TRD	Total	TRD
Berlin_L256	256x256	236±26	0±4	508±119	1±13
brc202d	481x530	1,389±597	4±11	9,451±1,128	9±14
empty-8-8	8x8	1±2	0±0	23±29	0±0
ht_chantry	141x162	95±150	1±6	856±2,180	0±6
maze-128-128-10	128x128	337±1,164	2±16	8,579±6,347	1±11
random-32-32-10	32x32	6±2	0±2	17±14	0±4
room-64-64-8	64x64	318±1,186	1±4	4,143±6,036	4±22

32-Neighbor Grid					
Map	Size	4 Agents		10 Agents	
		Total	TRD	Total	TRD
Berlin_L256	256x256	775±430	1±14	1,484±20	21±98
brc202d	481x530	1,929±405	24±46	3,902±6,053	131±205
empty-8-8	8x8	2,404±4,137	0±1	5,120±5,128	2±7
ht_chantry	141x162	1,980±3,198	6±11	6,836±7,102	11±27
maze-128-128-10	128x128	3,183±7,640	26±49	17,635±9,240	110±130
random-32-32-10	32x32	16±4	0±5	1,445±1,421	5±16
room-64-64-8	64x64	4,689±5,005	5±12	21,463±9,907	33±78

Table 3: Mean and standard deviation of total runtime and portion of runtime attributed to TRD duplicate detection in milliseconds.

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